Hamiltonian Monodromy: Dynamical Systems and Physics

Gabriela Gutierrez

Institut de Mathématiques de Bourgogne, Université de Bourgogne-Franche Comté

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Description of the system

- ▷ $\mathbb{R}^4 \rightsquigarrow (q,p) = ((x,y),(p_x,p_y))$
- ▷ Hamiltonian differential equation

  $H\colon \mathbb{R}^4 \to \mathbb{R}$ *s.t.* $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$

  Property:

  $\varphi(t)$

  $\frac{dH(\varphi(t))}{dt} = 0$ (*First integral*)

  Solutions contained in the level curves of H
- ▷ $K\colon \mathbb{R}^4 \to \mathbb{R}$ first integral independent of H

- Completely integrable system

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Action-Angle coordinates

Arnold-Liouville Theorem

The connected components of a regular compact fiber of $\mathcal{EM}$ are diffeomorphic to real tori $\mathbb{T}^2$. Moreover, the diffeomorphism is such that

$$\psi \colon U \to D^2 \times \mathbb{T}^2$$
$$(q, p) \mapsto (I, \phi)$$

and the differential equations transform into

$$\dot{I} = 0$$
$$\dot{\phi} = V(I)$$

A-A coordinates $\Rightarrow$ solutions, linear flow over invariant tori.

- Is this change of coordinates global?

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The Spherical Pendulum

- Video

◊ Differential equations
$q = (x, y, z) \in \mathbb{S}^2$, $p = (p_x, p_y, p_z) \in T_q\mathbb{S}^2$

$$\dot{\vec{q}} = \vec{p}$$
$$\dot{\vec{p}} = -\vec{e}_z - (p_x^2 + p_y^2 + p_z^2 - z)\vec{q}$$

◊ Hamiltonian (Energy)

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + z$$

◊ First integral independent of H

$$K = xp_y - yp_x$$

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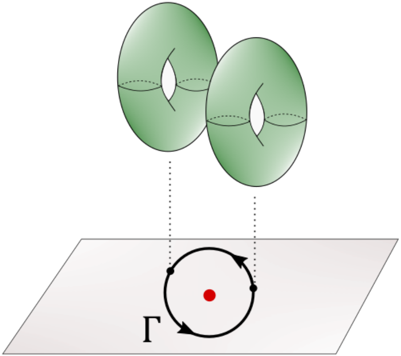
Hamiltonian Monodromy (HM)

- Diustermaat (1980): On global action-angle coordinates.



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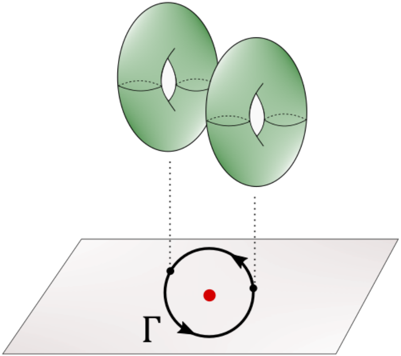

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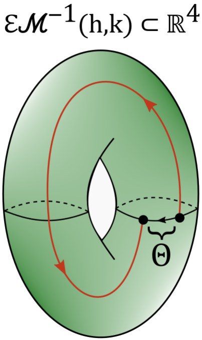
Symmetry under rotations associated to $K \Rightarrow \Theta(h, k)$.



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Rotation number Θ

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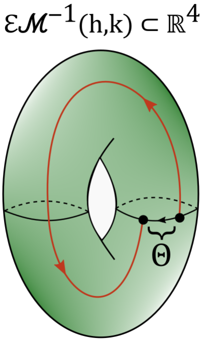

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Spectral Lax Pairs

$\mathbb{R}^4$

- ▷ Lax pair: L, M, 2×2 complex matrices time dependent s.t.

$$\dot{L} = [M, L] = ML - LM$$

- ▷ Spectral Lax pair: $L(\lambda)$, $M(\lambda)$, extra parameter $\lambda \in \mathbb{C}$
- ▷ Spectral curve: $0 = \det(L(\lambda) - \mu Id)$, does not depend on t

For this problem:

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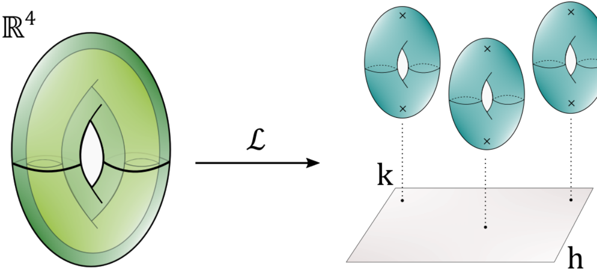
Hamiltonian monodromy via spectral Lax pairs



$$\Theta = \int_\gamma \eta \rightsquigarrow \textit{Picard-Lefschetz theory}$$

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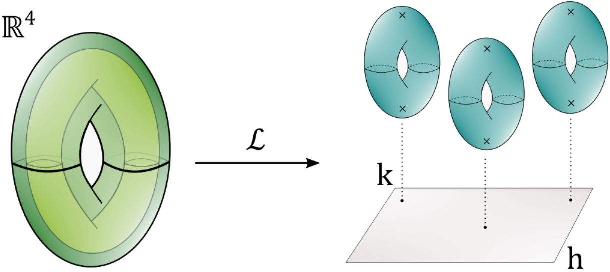

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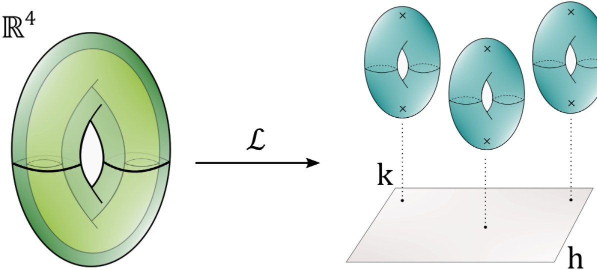

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Geometrical Theorem

Theorem [Mardesic, Sugny, G]

Consider the space (λ, μ) where $\mu^2 = Q_{h,k}(\lambda)$ and Q is a polynomial of degree 4 that has two non-real double roots for (h_0, k_0). Let ξ be a 1-form in this space which can be written as $\xi = c_1 \frac{\lambda}{d\lambda}\mu + c_2 \frac{d\lambda}{\mu}$, $(c_1, c_2) \in \mathbb{C}^2$. The variation of $\mathcal{I} = \int_\delta \xi$ when h and k vary along Γ is given by $\Delta_\Gamma \mathcal{I} = 2\pi i \operatorname{Res}(\xi, \infty)$. Moreover, this residue can be expressed as

$$\operatorname{Res}(\xi, \infty) = -\frac{c_1}{\sqrt{a_4}},$$

where a_4 is the leading coefficient of Q.

Monodromy Theorem

Corollary

Consider a completely integrable system in a neighborhood of an isolated critical value (h_0, k_0) with invariance under rotations associated to K and described by a spectral Lax pair with spectral curve $\mu^2 = Q_{h,k}(\lambda)$. Assume that the residue of η at infinity is $\frac{1}{i}$.
Under conditions given in the previous theorem, the monodromy matrix is

$$\mathcal{M}_\Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Example

Spherical Pendulum

- ◊ Isolated critical value: $(h, k) = (1, 0)$
- ◊ Spectral Lax pair
- ◊ Spectral curve: $\mu^2 = \lambda^4 - 2k\lambda^3 + 2h\lambda^2 + 1$
- ◊ 1-form: $\eta = \frac{i\lambda}{\mu} d\lambda$, residue $\frac{1}{i}$ at infinity

$$\mathcal{M}_\Gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

non trivial HM $\Rightarrow$ non global A-A coordinates

G. J. Gutierrez Guillen, P. Mardesic, D. Sugny, *Hamiltonian Monodromy via spectral Lax pairs*.

Thank you for the attention!